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# Duality in Two Capacitively Coupled Layered Arrays of Ultrasmall Josephson Junctions

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We consider the problem of two capacitively coupled Josephson junction arrays made of ultrasmall junctions. Each one of the arrays can be in the semiclassical or quantum regimes, depending on its physical parameter values. The former case is dominated by a Cooper-pair superfluid, while the quantum case is dominated by dynamic vortices leading to an insulating behavior. We first consider the limit when both arrays are in the semiclassical limit, and next the case when one array is quantum and the other semiclassical. We present WKB and mean-field theory results for the critical temperature of each array when both are in the semiclassical limit. When one array is in the semiclassical regime and the other one is in the quantum-fluctuations-dominated regime, we derive a duality transformation between the charge and vortex-dominated arrays that involves a gauge vector field which is proportional to the site coupling capacitance between the arrays. The system considered here has been fabricated and we make some predictions as to possible experimentally measurable quantities that could be compared with theory.

**KEY WORDS:** Superconductivity; macroscopic quantum phenomena; mesoscopic systems.

## I. INTRODUCTION

The subject treated in this paper relates to interesting quantum properties of Josephson junction arrays (JJA) made with ultrasmall junctions. Layered Josephson junction arrays have been the source of many theoretical and experimental studies in the last few years.[1] Recent advances in submicrometer technology have made it possible to fabricate relatively large arrays of ultrasmall superconductor-insulating-superconductor (SIS)

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Josephson junctions.[2-5] The areas of these junctions can vary from a few microns to submicron sizes. Under these circumstances the long range phase coherent properties of the JJA depend crucially on the interplay between the Josephson energy, $E_J$, and charging energy, $E_C$. Detailed experiments have been carried out, for example at Delft, that have produced a phase diagram of temperature vs the quantum parameter $\alpha \equiv E_C/E_J$.[2] We have calculated the $\alpha$ vs temperature phase diagram and we made[6,7] a direct successful comparisons to the experimental results.[2] Our results were obtained using a WKB-renormalization-group approach, plus a variational and quantum Monte Carlo (QMC) calculations.[6] An important QMC result is that there appears to be a low temperature QUantum fluctuation Induced Transition (QUIT) in this system.[8,6]

For the most part the experimental systems have been two-dimensional, but prototype quasi-three-dimensional samples have also been fabricated.[9] In this paper quasi–means two layers of JJA capacitively coupled at each lattice site. There are two dominant contributions to the charging energy in the type of junctions fabricated;[9] one due to the addition of a Cooper pair charge to a superconducting island given by $E_{C_s} = 2e^2/C_s$ with $C_s$ the self capacitance and $e$ the electronic charge, and the charging energy necessary to transfer a Cooper charge from one island to its nearest neighbor, given by $E_{C_m} = 2e^2/C_m$, with $C_m$ the mutual capacitance. In the Delft experiments, $C_s \sim 3 \times 10^{-17}$ F, and $C_m \sim 1 \times 10^{-15}$ F, which means that $C_m$ can be two orders of magnitude larger than $C_s$.

The phase diagram for one layered JJA has the following general characteristics. At low temperatures, for small $\alpha$ there is a superconducting phase in which the Cooper pair charges are delocalized, while for large $\alpha$ the system has delocalized vortices and the array is an insulator. There is a phase boundary that separates the superconducting to insulating regions. Now assume that we have two layered JJA capacitively coupled at each lattice site. This configuration is potentially quite interesting since, as mentioned above, each array can be in one of two extreme limits; one Cooper charge dominated and the other vortex dominated. Each array is now described by its quantum ratio $\alpha_i = E_{C_i}/E_{J_i}$ $(i = 1, 2)$. We can then imagine to have the two arrays in four possible configurations. When $\alpha_i \ll 1$, the $i$th array is dominated by localized vortex excitations, $V_i$, while the Copper pair excess charge excitations, $Q_i$, are in a superfluid or superconducting state. In the $\alpha_i \gg 1$ regime the array has the $Q_i$'s localized in an insulating state while the $V_i$'s are delocalized. We can also have the extreme cases when both arrays are semiclassical or in the quantum regimes.

In this paper we shall consider first the case where both arrays are in the semiclassical regime. We obtain an effective partition function that allows to calculate the change of each array critical temperatures. This type of

analysis was quite informative in the one-array problem. Next we move to consider the general case where we derive a Hamiltonian that is valid for all parameter regimes. This Hamiltonian is quite complex and difficult to analyze in full. Instead we consider the interesting case when one of the arrays is charge and the other vortex dominated. After a series of transformation we arrive at an effective Hamiltonian that exhibits interesting duality properties. We analyze the imaginary time dynamics of this Hamiltonian in the case where only one vortex in one array and one charge in the other are considered. Here we show that these two excitations interact via a gauge-like interaction proportional to the interaction capacitance between the arrays.

The outline of the paper is the following. In Section III the case where both arrays are in the semiclassical limit since in that case we can analyze in some detail the changes in the critical temperature of both arrays independently via a variational Mean Field analysis. In Section IV we consider the interesting limit when one array is semiclassical and the other quantum. In this case we can derive an effective action for the problem that allows a general analysis of the interaction of a vortex in one array with a charge in the other via an interaction term that has tile form of a minimal gauge coupling proportional to the interaction capacitance. The other extreme case where both arrays are quantum is not considered here since that case is harder to analyze. We conclude the paper with some conclusions in Section V.

In this paper I present some results of work done in collaboration with C. Rojas (related details can be found in ref. 6 and in his Ph.D. thesis[7]).

## II. THE MODEL

In this section we define the model that describes a quasi-three-dimensional array composed of two JJA layers coupled at each site by an ultrasmall capacitor. In our analysis we will have in mind the prototypical samples fabricated at Delft.[9] In these samples the size for each layer were $L_x = 230$, and $L_y = 60$. The typical parameter for the intra-layer mutual capacitance was $C_m \approx 2.3fF$, and the interplane local interaction capacitance was $C_{int} \approx 0.6fF$. The model consists of two planar arrays stacked on top of each other. The intra-array interaction between the superconducting islands in each array contains an electrostatic and a Josephson coupling. The two planes are only capacitively coupled. The model Hamiltonian is given by

$$\hat{H} = \frac{q}{2} \sum_{\vec{r}} \sum_{\mu} \hat{\mathbf{V}}_{\mu}(\vec{r})\, \hat{n}_{\mu}(\vec{r}) + F_1(\{\hat{\phi}_1\}) + F_2(\{\hat{\phi}_2\}) \tag{1}$$

where the index $\mu = 1, 2$ labels the two arrays. The operators $\hat{\phi}_\mu$, and $\hat{n}_\mu$, satisfy the commutation relations $[\hat{n}_\mu(\vec{r}_1), \hat{\phi}_{\mu'}(\vec{r}_2)] = -i\delta_{\vec{r}_1, \vec{r}_2}\delta\mu, \mu'$. The functions $F_\mu$ are the Josephson interaction terms,

$$F_\mu(\{\hat{\phi}\}) = E_J^{(\mu)} \sum_{\langle \vec{r}_1, \vec{r}_2 \rangle} (1 - \cos(\hat{\phi}_\mu(\vec{r}_1) - \hat{\phi}_\mu(\vec{r}_2))) \tag{2}$$

Here $E_J^{(\mu)}$ is the Josephson energy coupling constant, for the junctions in array $\mu$. $\mathbf{V}_\mu(\vec{r})$ is the electrostatic potential felt by the charges contained in the superconducting island located at $\vec{r}$ in array $\mu$. This potential is produced by all the other charges in both arrays, and it is obtained from the discrete Poisson equation

$$qn_1(\vec{r}) = C_{\mathrm{m}}^{(1)} \sum_{\vec{u}} [\mathbf{V}_1(\vec{r}) - \mathbf{V}_1(\vec{r} + \vec{u})] + C_{\mathrm{s}}^{(1)} \mathbf{V}_1(\vec{r}) + C_{\mathrm{int}}[\mathbf{V}_2(\vec{r}) - \mathbf{V}_1(\vec{r})] \tag{3}$$

Here the $\vec{u}$ summation is over nearest neighbors. In a uniform square lattice $\vec{u} = \{\pm\hat{x}, \pm\hat{y}\}$. The complementary equation for $n_2(\vec{r})$ is obtained by interchanging $(1 \leftrightarrow 2)$. In Eq. (3), $C_{\mathrm{s}}^{(\mu)}$ is the self capacitance of a superconducting island in array $\mu$. The previous equation can be written in the compact form,

$$qn_\mu(\vec{r}_1) = \sum_\nu \sum_{\vec{r}_2} \mathbf{C}_{\mu,\nu}(\vec{r}_1, \vec{r}_2)\, \mathbf{V}_\nu(\vec{r}_2) \tag{4}$$

where the capacitance supermatrix $\mathbf{C}_{\mu,\nu}$ is made of four blocks labeled by $\mu, \nu = 1, 2$,

$$\mathbf{C}_{\mu,\nu}(\vec{r}_1, \vec{r}_2) = \begin{cases} (C_{\mathrm{s}}^{(\mu)} + zC_{\mathrm{m}}^{(\mu)} + C_{\mathrm{int}}), & \text{if } \mu = \nu \text{ and } \vec{r}_1 = \vec{r}_2, \\ -C_{\mathrm{m}}^{(\mu)}, & \text{if } \mu = \nu \text{ and } \vec{r}_1 = \vec{r}_2 \pm \vec{u} \\ -C_{\mathrm{int}}, & \text{if } \mu \neq \nu \text{ and } \vec{r}_1 = \vec{r}_2, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

The diagonal blocks of this matrix are the intra-array capacitance matrices. We will use the following notation for them

$$\mathbf{C}_\mu = \mathbf{C}_{\mu,\nu} \tag{6}$$

The off-diagonal parts of the $\mathbf{C}_{\mu,\nu}$ supermatrix are given by block matrices proportional to the identity matrix $-C_{\mathrm{int}}\mathbf{I}_{N,N}$, with $N$ the linear size of

the arrays. The inverse matrix of Eq. (5), $\tilde{\mathbf{C}}$, is obtained from solving the equation

$$\sum_{\nu} \sum_{\vec{r}_2} \tilde{\mathbf{C}}_{\mu,\nu}(\vec{r}_1, \vec{r}_2)\, \mathbf{C}_{\nu,\rho}(\vec{r}_2, \vec{r}_3) = \delta_{\mu,\rho}\, \delta_{\vec{r}_1,\vec{r}_3} \tag{7}$$

The explicit components of the inverse matrix are given by

$$\tilde{\mathbf{C}}_{1,1} = \mathbf{C}_1^{-1}[\mathbf{I} - C_{\text{int}}^2 \mathbf{C}_2^{-1}\mathbf{C}_1^{-1}]^{-1} \tag{8}$$

$$\tilde{\mathbf{C}}_{2,2} = \mathbf{C}_2^{-1}[\mathbf{I} - C_{\text{int}}^2 \mathbf{C}_1^{-1}\mathbf{C}_2^{-1}]^{-1} \tag{9}$$

$$\tilde{\mathbf{C}}_{1,2} = C_{\text{int}} \mathbf{C}_1^{-1}\mathbf{C}_2^{-1}[\mathbf{I} - C_{\text{int}}^2 \mathbf{C}_1^{-1}\mathbf{C}_2^{-1}]^{-1} \tag{10}$$

$$\tilde{\mathbf{C}}_{2,1} = C_{\text{int}} \mathbf{C}_2^{-1}\mathbf{C}_1^{-1}[\mathbf{I} - C_{\text{int}}^2 \mathbf{C}_2^{-1}\mathbf{C}_1^{-1}]^{-1} \tag{11}$$

For a uniform array an explicit expression for this matrix can be found using a Fourier representation of the matrices.

Finally, using Eqs. (1), (4), and (7), we can write down the model Hamiltonian studied in this paper as

$$\hat{H} = \frac{q^2}{2} \sum_{\vec{r}_1,\vec{r}_2} \sum_{\mu,\nu} \hat{n}_\mu(\vec{r}_1)\, \tilde{\mathbf{C}}_{\mu,\nu}(\vec{r}_1, \vec{r}_2)\, \hat{n}_\nu(\vec{r}_2) + F_1(\{\hat{\phi}_1\}) + F_2(\{\hat{\phi}_2\}) \tag{12}$$

## III. SEMICLASSICAL CAPACITIVELY COUPLED ARRAYS

In this section we begin our analysis of the quasi-three dimensional JJA model defined by Eq. (12), when the parameters in both arrays are in the small $\alpha$ semiclassical regime. Here we estimate the change in the critical temperature, $T_c$, for each array as a function of $C_{\text{int}}$. We use a WKB semiclassical expansion valid for small $\alpha$ values, as we did in the single array problem.[8, 6] In this limit we first obtain an effective action that we then analyze within a variational Mean Field theory (MFT) approach. In the prototypical Delft samples, they had small ratio values for $(C_{\text{int}}/C_s)$. To have a consistent semiclassical expansion in $C_{\text{int}}$, as follows from looking at Eqs. (8)–(11), we need to carry out the expansion at least to second order in $\tilde{\mathbf{C}}$, which is equivalent to doing the expansion up to second order in $q^2$. After a long, but direct, calculation we obtain the effective semiclassical partition function,

$$Z_{SC} = \int d\bar{\phi} \exp\{-S_{\text{eff}}[\bar{\phi}]\} \tag{13}$$

with the effective semiclassical action up to second order given by

$$S_{\text{eff}}[\phi] = \beta \sum_{\mu} \left\{ 1 - \frac{\beta q^2}{12} [\tilde{\mathbf{C}}_{\mu,\mu}(|\vec{0}|) - \tilde{\mathbf{C}}_{\mu,\mu}(|\vec{u}|)] + \frac{(q^2\beta)^2}{1152} [\tilde{\mathbf{C}}_{\mu,\mu}(|\vec{0}|) - \tilde{\mathbf{C}}_{\mu,\mu}(|\vec{u}|)]^2 \right\} F_{\mu}(\phi_{\mu}) + \frac{(q^2\beta)^2 \beta^2}{720} \mathbf{Tr} \left\{ \sum_{\mu,\nu} \tilde{\mathbf{C}}_{\mu,\nu} \frac{\partial^2 F_{\nu}}{\partial \phi_{\nu}^2} \tilde{\mathbf{C}}_{\nu,\mu} \frac{\partial^2 F_{\mu}}{\partial \phi_{\mu}^2} \right\} \tag{14}$$

As a check, we note that we recover the one-layer results if we keep terms up to first order in $q^2$. We find that the superconducting state is stable at low temperatures, because the second-order contribution in $q^2$ has a negative sign compared to the first-order contribution to the effective action. Notice that higher orders in $q^2$ coincide with higher orders in $\beta$. This conclusion does not, by itself, eliminate the possibility of having a *QUIT* in the two layered problem.

Equation (14) does not have a simple form as in the one JJA layer first-order expansion in $q^2$. The third term in Eq. (14) has nonlocal interactions, which makes a direct calculation of the critical temperature in general complicated. To estimate the change in the critical temperatures as a function of $C_{\text{int}}$, we have performed a couple of distinct MFT variational calculations for the partition function given in Eq. (14).

For the general variational calculation first we split the effective action into two parts

$$S_{\text{eff}}[\phi_1, \phi_2] = S_0^{(1)}[\phi_1] + S_0^{(2)}[\phi_2] + (S_{\text{eff}}[\phi_1, \phi_2] - S_0^{(1)}[\phi_1] - S_0^{(2)}[\phi_2]) \tag{15}$$

which gives the exact semiclassical expansion for the partition function

$$Z = Z_0^{(1)} Z_0^{(2)} \langle \exp\{ -S_{\text{eff}}[\phi_1, \phi_2] + S_0^{(1)}[\phi_1] + S_0^{(2)}[\phi_2] \} \rangle_0 \tag{16}$$

$$Z_0^{(\mu)} = \int d\phi_{\mu} \exp\{ -S_0[\phi_{\mu}] \} \tag{17}$$

The average $\langle\ \rangle_0$ is defined as

$$\langle A \rangle_0 = \frac{1}{Z_0^{(1)} Z_0^{(2)}} \int d\phi_1\, d\phi_2\, A(\phi_1, \phi_2) \exp\{ -S_0^{(1)}[\phi_1] - S_0^{(2)}[\phi_2] \} \tag{18}$$

We now use the variational inequality $\langle \exp\{A\} \rangle \leqslant \exp\{\langle A \rangle\}$ to write

$$Z \leqslant Z_0^{(1)} Z_0^{(2)} \exp\{\langle -S_{\text{eff}}[\phi_1, \phi_2] + S_0^{(1)}[\phi_1] + S_0^{(2)}[\phi_2] \rangle_0\} \tag{19}$$

The corresponding variational free energy is then

$$\beta F_{\text{var}} = -\ln Z_0^{(1)} - \ln Z_0^{(2)} - \langle -S_{\text{eff}}[\phi_1, \phi_2] + S_0^{(1)}[\phi_1] + S_0^{(2)}[\phi_2] \rangle_0 \tag{20}$$

We next need to specify the functions $S_0^{(\mu)}$. They can be any general functions, but here we restrict them to be functions of only one phase variable for each layer. These functions must be chosen so that we can carry out some or all of the integrations in Eq. (20). We introduce variational parameters in the trial actions that are determined by requiring that they minimize $\beta F_{\text{var}}$. We have used two different choices for $S_0^{(\mu)}$.

## A. First Variational Calculation

The first choice decouples all the phases in both arrays

$$S_0^{(\mu)}[\phi_\mu] = -\gamma_\mu \sum_{\vec{r}} \cos(\phi_\mu(\vec{r})) \tag{21}$$

with the two $\gamma_\mu$'s the variational parameters.[10] The advantage of this form for the trial action is that all the integrals can be analytically computed or they can be expressed as simpler one-dimensional integrals. For the classical 2-D XY model it is known that this variational choice grossly overestimate the critical temperature.[10] This approximation, nonetheless, gives good qualitative results for the critical temperatures for both JJA layers. Here we are mostly interested in general trends so, for simplicity, we only study the case when both JJA layers have the same parameter values. After a lengthy but direct calculation, the critical temperature equation for one of arrays is obtained from solving the equation

$$D_4 x^4 + D_3 x^3 + D_2 x^2 + D_1 x + D_0 = 0 \tag{22}$$

were we found

$$x = \beta_c E_J, \qquad \alpha = e^2/(2C_s E_J), \qquad D_0 = -1, \qquad D_1 = 2 \tag{23}$$

$$D_2 = -(4/3)\,\alpha K_1, \qquad D_3 = \alpha^2 K_1^2/9, \qquad D_4 = 4\alpha^2(K_2/2 - K_3)/45 \tag{24}$$

$K_1$, $K_2$ and $K_3$ are complicated functions of the ratios $(C_{\text{int}}/C_s)$ and $(C/C_s)$. They can be computed in general in terms of summations over Fourier modes. Here we are only interested in a general trend of $T_c$ as a function

of $C_{int}$, and we further simplify the problem by considering the self-capacitive limit, i.e., $C_m = 0$. In this case, the $K$ functions can be fully computed giving

$$K_1 = \left(\frac{1 + C_{int}/C_s}{1 + 2C_{int}C_s}\right), \qquad K_2 = 28K_1^2, \qquad \text{and} \qquad K_3 = 8K_1^2 \tag{25}$$

We can now interpret the effect of the inter-plane capacitance as a resealing of the single layer quantum parameter $\alpha$, namely

$$\alpha_{eff} = \left(\frac{1 + C_{int}/C_s}{1 + 2C_{int}/C_s}\right)\alpha \tag{26}$$

The first conclusion we draw from this result is that, in the semiclassical approximation, the inter-plane capacitance makes the system *less quantum mechanical.* The critical temperature increases as $(C_{int}/C_s)$ increases up to an asymptotic plateau. To further check this result we have also performed quantum Monte Carlo calculations that confirmed our analytic results.

Note that here we only presented the case where both arrays are equal and the mutual capacitance is zero. We have done so due to the simplicity of the analytic results. It is not much harder, within this approximation, to numerically calculate $T_c$ for the more general cases. The general result leads to the same qualitative conclusion; The increase in the inter-plane capacitance raises the critical temperature for both arrays.

From Eq. (26) we can find $T_c$ up to second order in the effective quantum parameter $\alpha_{eff}$ giving

$$\left(\frac{k_B T_c}{E_J}\right) = \left(\frac{k_B T_c^{(0)}}{E_J}\right) - \left(\frac{2}{3}\right)\alpha_{eff} + \left(\frac{189}{180}\right)\alpha_{eff}^2 + O(\alpha_{eff}^3) \tag{27}$$

where the variational result for the classical 2-D XY model is $(k_B T_c^{(0)}/E_J) = 2$. It is evident that this approximation overestimated the critical temperature. Surprisingly, it gives a very good estimate of the first two correction values for the one array problem.[8]

## B. Second Variational Calculation

In the previous subsection we used Eq. (21) to perform the MFT variational calculation. From the form of Eq. (20) it is clear that it is not necessary to use a trial action that decouples all the phases in both arrays. We now want to use a better trial action that gives better 2-D XY model classical results. This has the advantage that the classical limit for each array is by construction exact. The disadvantage of this choice is that we

need to evaluate a nonlocal average over the 2-D XY model classical Hamiltonian, for which we have to approximately evaluate an infinite lattice sum. Again, we restrict the calculation to the $C_m = 0$ limit, although we could do the numerical calculation for the full model.

The starting point of this scheme is to use the trial action

$$S_0^{(\mu)}[\phi_\mu] = -\beta_\mu \sum_{\langle \vec{r}_1, \vec{r}_2 \rangle} \cos(\phi_\mu(\vec{r}_1) - \phi_\mu(\vec{r}_2)) \tag{28}$$

with $\beta_1$ and $\beta_2$ the variational parameters needed to calculate the variational free energy in Eq. (20) from Eq. (14). We need to evaluate the following averages

$$\langle F_\mu \rangle_{XY} = E_J^{(\mu)} \sum_{\langle \vec{r}_1, \vec{r}_2 \rangle} (1 - \langle \cos(\phi_\mu(\vec{r}_1) - \phi_\mu(\vec{r}_2)) \rangle_{XY}) \tag{29}$$

$$\left\langle \tilde{\mathbf{C}}_{\mu,\nu} \frac{\partial^2 F_\nu}{\partial \phi_\nu^2} \tilde{\mathbf{C}}_{\nu,\mu} \frac{\partial^2 F_\mu}{\partial \phi_\mu^2} \right\rangle_{XY} \sim \langle \cos(\phi_\mu(\vec{r}_1) - \phi_\mu(\vec{r}_2)) \cos(\phi_\nu(\vec{r}_3) - \phi_\nu(\vec{r}_4)) \rangle_{XY} \tag{30}$$

The average in Eq. (29) is simple, since we only need to perform the average when $\vec{r}_1$ and $\vec{r}_2$ are nearest neighbors. Here we are interested in periodic and symmetric arrays. It is then convenient to define and evaluate the following short range correlation function

$$g_0(\beta) = \langle \cos(\phi(\vec{r}) - \phi(\vec{r} + \vec{u}) \rangle_{XY} \tag{31}$$

with $\beta$ the inverse temperature of the classical 2-D XY model. This function can not be calculated exactly, but it can be evaluated using a Monte Carlo calculation or by matching a low to a high-temperature expansion for the classical 2-D XY model.

The average in Eq. (30) is more complicated. First, if $\mu \neq \nu$, due to the intra-plane independence, the average can be reduced to finding two values of $g_0(\beta)$. If $\mu = \nu$, the problem is intractable for general $\vec{r}_1$ and $\vec{r}_3$. This is exactly the problem we would encounter if we want to make a calculation for a general full capacitance matrix. On the other hand if we choose $C_m = 0$, the problem is simplified. In this case, only $g_0(\beta)$ and the following functions

$$g_1(\beta) = \langle [\cos(\phi(\vec{r}) - \phi(\vec{r} + \hat{x}))]^2 \rangle_{XY} \tag{32}$$

$$g_2(\beta) = \langle \cos(\phi(\vec{r}) - \phi(\vec{r} + \hat{x})) \cos(\phi(\vec{r}) - \phi(\vec{r} + \hat{y})) \rangle_{XY} \tag{33}$$

$$g_3(\beta) = 2 \left\{ \frac{2(dg_1/d\beta) + (dg_2/d\beta)}{dg_0/d\beta} \right\} \tag{34}$$

need to be known. These function can be calculated using a Monte Carlo calculation for the classical 2-D XY model. Using all these functions, we can evaluate the variational free energy $F_{\text{var}}(\beta_1, \beta_2)$. The parameters $\beta_1$ and $\beta_2$ are found by imposing the condition that they minimize the free energy. This condition reads

$$\frac{\partial F_{\text{var}}}{\partial \beta_1} = 0, \qquad \frac{\partial F_{\text{var}}}{\partial \beta_2} = 0 \tag{35}$$

The explicit equation for $\beta_1$ obtained from Eq. (35) is

$$\beta_1 = (\beta E_J^{(1)}) \left\{ 1 - \frac{(\beta E_J^{(1)})}{3} \alpha_{\text{eff}}^{(1)} + \frac{(\beta E_J^{(1)})^2}{18} \alpha_{\text{eff}}^{(1)^2} \right\} + \frac{4(\beta E_J^{(1)})^4}{45} \left\{ \alpha_{\text{eff}}^{(1)^2} g_3(\beta_1) + 16 \left( \frac{E_J^{(2)}}{E_J^{(1)}} \right)^2 \alpha_{\text{eff}}^{(1)} \alpha_{\text{eff}}^{(2)} \left( \frac{C_{\text{int}}/C_s^{(1)}}{1 + C_{\text{int}}/C_s^{(1)}} \right) \left( \frac{C_{\text{int}}/C_s^{(2)}}{1 + C_{\text{int}}/C_s^{(2)}} \right) g_0(\beta_2) \right\} \tag{36}$$

A similar equation can be written down for $\beta_2$. In writing this equation we have used the following definition for the effective quantum parameter for the arrays

$$\alpha_{\text{eff}}^{(1)} = \frac{e^2}{2C_s^{(1)} E_J^{(1)}} \left( \frac{1 + C_{\text{int}}/C_s^{(2)}}{1 + C_{\text{int}}/C_s^{(1)} + C_{\text{int}}/C_s^{(2)}} \right) \tag{37}$$

We are therefore left with the following set of self-consistent equations for the two variational parameters

$$\beta_1 = G_1(\beta, \beta_1, \beta_2), \qquad \beta_2 = G_2(\beta, \beta_1, \beta_2) \tag{38}$$

The functions $G_1$ and $G_2$ can be identified from Eq. (36). We set $\beta_\mu = \beta_c^{XY}$ as the condition to find the critical temperature of the array $\mu$, i.e., we identify the value of $\beta_\mu$, with an effective inverse temperature for the array $\mu$. To find the critical temperature for array 1 we solve the following set of equations

$$\beta_c^{XY} = G_1(\beta_c^{(1)}, \beta_c^{XY}, \beta_2), \qquad \beta_2 = G_2(\beta_c^{(1)}, \beta_c^{XY}, \beta_2) \tag{39}$$

These equations are not difficult to solve. First, given a value of $\beta$ we first solve for $\beta_2$ using Eq. (39). Note that this equation can be written in the following form

$$\beta_2 = A + B g_3(\beta_2) \tag{40}$$

We can show from Eq. (36) that the quantities $A$ and $B$ in this equation are not functions of $\beta_2$. Moreover, solving this equation using a fixed point search method gives $\beta_2(\beta_c^{(1)})$, that can then be introduced into Eq. (39). We are now left with a one variable equation.

The case when both arrays have the same parameters is again easier to solve, since the transcendental Eqs. (39) reduce to just one polynomial equation

$$R_4 x^4 + R_3 x^3 + R_2 x^2 + R_1 x + R_0 = 0 \tag{41}$$

where $\beta_c^{(1)} = \beta_c^{(2)}$ and

$$R_0 = -\beta_c^{XY}, \qquad R_1 = 1, \qquad R_2 = -\tfrac{2}{3}\alpha_{\text{eff}}, \qquad R_3 = \tfrac{1}{18}\alpha_{\text{eff}}^2 \tag{42}$$

$$R_4 = \frac{4}{45}\alpha_{\text{eff}}^2 \left[ 3.6 + 9.6 \left( \frac{C_{\text{int}}/C_s}{1 + C_{\text{int}}/C_s} \right)^2 \right] \tag{43}$$

We have used Eq. (26) to define $\alpha_{\text{eff}}$ and found the value of $g_3(\beta_c^{XY})$ using Monte Carlo calculations to be

$$\beta_c^{XY} \approx 1.1186, \qquad g_3(\beta_c^{XY}) \approx 3.6 \tag{44}$$

The result of this calculation gives the same result as in Eq. (27) for the first order correction in $\alpha$, but now the $\alpha = 0$ limit gives the correct classical value for $T_c/k_B E_J = 1/\beta_c^{XY}$. Here the general result is qualitatively the same; an increase in the interaction capacitance $C_{\text{int}}$ results in a decrease of the effective quantum parameter.

## IV. DUALITY IN TWO CAPACITIVELY COUPLED JJA

In the previous section we studied the specific changes in the individual critical temperature of two JJA, when both were in the semiclassical parameter regime. In this section we analytically consider the interesting case where one array is in the semiclassical regime and the other in the quantum one. In the semiclassical array, vortices are localized while the Cooper pair charge fluctuations are mobile in the superfluid phase. This happens for small $\alpha$ values. In the quantum JJA regime, large $\alpha$ the vortices are mobile while the charges are localized. It is not possible to have both vortices and charges simultaneously localized or mobile since this is forbidden by the Heisenberg uncertainty principle (which has been shown at work in recent experiments in a small Josephson array system[11]). The interaction between vortices and charges has a minimal coupling form, with constant strength and it is sharply localized, i.e., a vortex and a Cooper pair only interact if they are located at the same point in the array.

By considering two arrays with one vortex dominated and the other charge dominated we can have them both interacting via the coupling capacitance between the arrays. A related analysis of the coupled array system was considered in ref. 15 by us and an alternative and complementary analysis was also presented in ref. 16.

In this section we will carry out the two array analysis extending techniques developed for the study of one array.[12] Here we shall consider first the one-array component of the Hamiltonian given in Eq. (12) and later its Villain approximation.[13] we briefly mention the one JJA calculational approach, since the extension to the two-array case follows from this analysis. This is true only because the arrays are electrostatically coupled.

The Hamiltonian for one array reads

$$\hat{H}=\frac{q^2}{2}\sum_{\vec{r}_1,\vec{r}_2}\hat{n}(\vec{r}_1)\,\mathbf{C}^{-1}(\vec{r}_1,\vec{r}_2)\,\hat{n}(\vec{r}_2)+E_J\sum_{\langle\vec{r}_1,\vec{r}_2\rangle}[1-\cos(\hat{\phi}(\vec{r}_1)-\hat{\phi}(\vec{r}_2))] \tag{45}$$

where $\mathbf{C}(\vec{r}_1,\vec{r}_2)$ is the one-array capacitance. The partition function for this Hamiltonian can be written in the path integral form

$$\begin{aligned}Z=\prod_{\vec{r}_1}\prod_{\tau}\sum_{\{n(\tau,\vec{r})\}}\int_0^{2\pi}\frac{d\phi(\tau,\vec{r})}{2\pi}\exp\Bigg\{-\int_0^{\beta\hbar}d\tau\\ \times\Bigg[\frac{q^2}{2}\sum_{\vec{r}_1,\vec{r}_2}n(\tau,\vec{r}_1)\,\mathbf{C}^{-1}(\vec{r}_1,\vec{r}_2)\,n(\tau,\vec{r}_2)\\ +i\sum_{\vec{r}}n(\tau,\vec{r})\frac{d\phi}{d\tau}(\tau,\vec{r})+E_J\sum_{\langle\vec{r}_1,\vec{r}_2\rangle}(1-\cos\phi_{\vec{r}_1,\vec{r}_2}(\tau))\Bigg]\Bigg\}\end{aligned} \tag{46}$$

Here we denote $\phi_{\vec{r}_1,\vec{r}_2}(\tau)=\phi(\tau,\vec{r}_1)-\phi(\tau,\vec{r}_2)$, and write the imaginary time summation as an integral. To integrate over the $\phi$'s, we need to introduce an additional set of variables. This is done by writing the Boltzmann factor as a Fourier series using the Poisson summation representation,[13]

$$\exp\{-\lambda(1-\cos\psi)\}=\sum_{m=-\infty}^{\infty}f_m(\lambda)\exp\{i\psi m\} \tag{47}$$

$$\begin{aligned}f_m(\lambda)&=\int_0^{2\pi}\frac{d\psi}{2\pi}\exp\{-\lambda(1-\cos\psi)\}\exp\{-i\psi m\}\\ &=\exp\{-\lambda\}\,I_m(\lambda)\end{aligned} \tag{48}$$

Here $I_m(\lambda)$ is the modified Bessel function. The integral in Eq. (48) gives a convenient way to extract the asymptotics for the small and large $\lambda$. For small $\lambda$, the Taylor series expansion of $I_m(\lambda)$ around $\lambda=0$ gives us the

leading expansion terms of $f_m(\lambda)$. For large $\lambda$, a steepest descent calculation yields the leading term. In these two asymptotic limits we have

$$f_m(\lambda) \approx \begin{cases} (\lambda/2)^2/m!, & \text{if } \lambda \ll 1 \\ \exp\{-m^2/2\lambda\}/\sqrt{2\pi\lambda}, & \text{if } \lambda \gg 1 \end{cases} \tag{49}$$

If we use the $\lambda \gg 1$ result in Eq. (47), we can write

$$\exp\{-\lambda(1-\cos\psi)\} \approx \frac{1}{\sqrt{2\pi\lambda}} \sum_{m=-\infty}^{\infty} \exp\{-m^2/2\lambda + i\psi m\} \tag{50}$$

Using Eq. (47) in Eq. (46) we have new summation link variables $m_\nu(\tau, \vec{r})$ between the nodes of the lattice. This is repeated for each of the imaginary time planes. The subindex $\nu$ is a vector that denotes the orientation of the links so that we can write

$$\exp\{-\varepsilon E_J[1-\cos(\phi(\tau, \vec{r}+\hat{\nu}) - \phi(\tau, \vec{r}))]\}$$
$$\approx \left(\frac{1}{2\pi\varepsilon E_J}\right)^{L_\tau L_x L_y} \sum_{m_\nu(\tau, \vec{r})} \exp\left\{-\frac{1}{2\varepsilon E_J} m_\nu(\tau, \vec{r})^2 + i m_\nu(\tau, \vec{r})\, \Delta_\nu \phi(\tau, \vec{r})\right\} \tag{51}$$

Here we have discretized the imaginary time interval with $\varepsilon = \beta\hbar/L_\tau$, and the lattice derivative is $\Delta_\nu f(\vec{r}) = f(\vec{r}+\hat{\nu}) - f(\vec{r})$. After the integrations over the $\phi$'s the result can be written in terms of the integer $n$ and $m$ variables as

$$Z \approx \sum_{\{n(\tau, \vec{r})\}} \sum_{\{\vec{m}(\tau, \vec{r})\}} \exp\left\{-\sum_\tau \left[\frac{\varepsilon Q^2}{2} \sum_{\langle \vec{r}_1, \vec{r}_2 \rangle} n(\tau, \vec{r}_1)\, \mathbf{C}^{-1}(\vec{r}_1, \vec{r}_2)\, n(\tau, \vec{r}_2) + \frac{1}{2\varepsilon E_j} \sum_{\vec{r}} |\vec{m}(\tau, \vec{r})|^2\right]\right\} \tag{52}$$

After the integration over the $\phi$'s we get a set of constraints over the $n$ and $m$ values. These constraints can be written as discrete continuity equations that are satisfied at each node of the array,

$$\vec{\Delta} \cdot \vec{m}(\tau, \vec{r}) + \Delta_\tau n(\tau, \vec{r}) = 0 \tag{53}$$

These constraint equations can be solved in several waifs. For example, the pair $(n, \vec{m})$ can be expressed in terms of a three-vector $\mathscr{K} = (n, \vec{m})$,[14] so that Eq. (53) becomes

$$\Delta_\nu \mathscr{K}^\nu = 0 \tag{54}$$

with the discrete gradient, $\Delta_\nu = (\Delta_\tau, \vec{\Delta})$. From Eq. (54) $\mathscr{K}$ can be expressed as the curl of a gauge field, i.e., $\mathscr{K}^\mu = \varepsilon^{\mu\nu\rho}\Delta_\nu \mathscr{A}_\rho$, with $\varepsilon^{\mu\nu\rho}$ the usual fully antisymmetric tensor. Substituting this result into Eq. (52) we get an effective action over the gauge field $\mathscr{A}$, which resolves the constraints over the summations.

Our solution to the constraint it Eq. (53) is different from the one used in ref. 12 where they wanted to preserve the n variables. The solution to our constraint equation will have a particular solution plus an homogeneous solution. Note that Eq. (53), written in this form, resembles one of Maxwell's equation that connect the divergence of the electric field to the charge density. The particular solution to this equation contains the gradient of a line integral, that can then be solved using a discrete line integral operator. The solution can be formally written as

$$m^\mu(\tau, \vec{r}) = -e^\mu(\hat{e}\cdot\vec{\Delta})^{-1}\Delta_\tau n(\tau, \vec{r}) + \varepsilon^{\mu\nu}\Delta_\nu A(\tau, \vec{r}) \tag{55}$$

with $A(\tau, \vec{r})$ another integer gauge field. To obtain the partition function we need to perform summations over this field. The first term in the last equation represents a discrete line integral, which is better calculated in a Fourier representation. After substituting Eq. (55) into Eq. (52) we obtain an expression for an effective action as functions of the $n$'s and the $A$'s. The partition function is now obtained by summing over these variables, but now without any constraint. We perform the summation over the $A$'s again with the help of the Poisson summation formula. After introducing a new set of integer $v$ variables, the summation over the $A$'s can now be done since the integrals left to calculate are Gaussian and the $A$'s are unconstrained fields. The final result is

$$Z = \sum_{\{n\}} \sum_{\{v\}} \exp[-S_{\text{eff}}(n, v)] \tag{56}$$

$$\begin{aligned} S_{\text{eff}}(n, v) = \sum_{\vec{r}_1, \vec{r}_2, \tau} \Bigg[ & \frac{q^2\varepsilon}{2\pi} n(\tau, \vec{r}_1)\, \mathbf{C}(\vec{r}_1, \vec{r}_2)\, n(\tau, \vec{r}_2) \\ & + \pi\varepsilon E_J v(\tau, \vec{r}_1)\, \mathbf{G}(\vec{r}_1, \vec{r}_2)\, v(\tau, \vec{r}_2) + in(\tau, \vec{r}_1)\, \boldsymbol{\Theta}(\vec{r}_1, \vec{r}_2)\, \Delta_\tau v(\tau, \vec{r}_2) \\ & + \frac{1}{4\pi\varepsilon E_J} \Delta_\tau n(\tau, \vec{r}_1)\, \mathbf{G}(\vec{r}_1, \vec{r}_2)\, \Delta_\tau n(\tau, \vec{r}_2) \Bigg] \end{aligned} \tag{57}$$

where we defined

$$\mathbf{G}(\vec{r}_1, \vec{r}_2) \approx \ln|\vec{r}_1 - \vec{r}_2| \qquad \text{and} \qquad \boldsymbol{\Theta}(\vec{r}_1, \vec{r}_2) \approx \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \tag{58}$$

Equation (57) is an effective action for two coupled imaginary time Coulomb gases. This equation is valid for all parameter ranges. When $E_J$ is large, the last term in this equation is small and the time derivatives of the charges are soft with the $n$'s having strong fluctuations. In this limit the $v$'s dominate and the $n$'s are not well defined. We will call this a vortex-dominated regime. When $E_J$ is small, the last term in the effective action is large and it makes the time derivatives of the charges well defined. In this limit the $v$'s are not well defined and the state is charge-dominated, with the charges described by an effective continuous Gaussian model. After integrating the continuous variables we obtain an effective action for the vortex integer conjugate variables.

One important aspect of the one-array derivation of Eq. (57) is that it did not involve the charging energy part. This means that when we carry out the two-array calculations we only need to see that its effective action can be written down immediately from just repeating the one array calculation; we only need to add the essential extra charging energy term that couples the two arrays. The two-array equivalent equation to Eq. (57) is then

$$
\begin{aligned}
&S_{\text{eff}}(n^{(1)}, v^{(1)}; n^{(2)}, v^{(2)}) \\
&\quad = \sum_{\vec{r}_1, \vec{r}_2, \tau} \Bigg[ \frac{q^2 \varepsilon}{2\pi} n^{(1)}(\tau, \vec{r}_1)\, \tilde{\mathbf{C}}_{1,1}(\vec{r}_1, \vec{r}_2)\, n^{(1)}(\tau, \vec{r}_2) \\
&\qquad + \pi \varepsilon E_J^{(1)} v^{(1)}(\tau, \vec{r}_1)\, \mathbf{G}(\vec{r}_1, \vec{r}_2)\, v^{(1)}(\tau, \vec{r}_2) \\
&\qquad + i n^{(1)}(\tau, \vec{r}_1)\, \boldsymbol{\Theta}(\vec{r}_1, \vec{r}_2)\, \Delta_\tau v^{(1)}(\tau, \vec{r}_2) \\
&\qquad + \frac{1}{4\pi \varepsilon E_J^{(1)}} \Delta_\tau n^{(1)}(\tau, \vec{r}_1)\, \mathbf{G}(\vec{r}_1, \vec{r}_2)\, \Delta_\tau n^{(1)}(\tau, \vec{r}_2) \Bigg] \\
&\qquad + \sum_{\vec{r}_1, \vec{r}_2, \tau} \Bigg[ \frac{q^2 \varepsilon}{2\pi} n^{(2)}(\tau, \vec{r}_1)\, \tilde{\mathbf{C}}_{2,2}(\vec{r}_1, \vec{r}_2)\, n^{(2)}(\tau, \vec{r}_2) \\
&\qquad + \pi \varepsilon E_J^{(2)} v^{(2)}(\tau, \vec{r}_1)\, \mathbf{G}(\vec{r}_1, \vec{r}_2)\, v^{(2)}(\tau, \vec{r}_2) \\
&\qquad + i n^{(2)}(\tau, \vec{r}_1)\, \boldsymbol{\Theta}(\vec{r}_1, \vec{r}_2)\, \Delta_\tau v^{(2)}(\tau, \vec{r}_2) \\
&\qquad + \frac{1}{4\pi \varepsilon E_J^{(2)}} \Delta_\tau n^{(2)}(\tau, \vec{r}_1)\, \mathbf{G}(\vec{r}_1, \vec{r}_2)\, \Delta_\tau n^{(2)}(\tau, \vec{r}_2) \Bigg] \\
&\qquad + \sum_{\vec{r}_1, \vec{r}_2, \tau} \Bigg[ \frac{q^2 \varepsilon}{\pi} n^{(1)}(\tau, \vec{r}_1)\, \tilde{\mathbf{C}}_{1,2}(\vec{r}_1, \vec{r}_2)\, n^{(2)}(\tau, \vec{r}_2) \Bigg]
\end{aligned}
\tag{59}
$$

where we used the definition of $\tilde{\mathbf{C}}$ given in Eq. (7).

In this section we shall consider the interesting case where one of the arrays is in the semiclassical regime (the vortex-dominated state) and the other is in the full quantum regime (i.e., the charge-dominated state). we want in particular to study the interaction between vortices in one array and charges in the other. We assume, as in experiment, that the arrays are dominated by the mutual capacitance between nearest neighbors. We take array 1 vortex-dominated and array 2 charge-dominated, i.e.,

$$E_{C_m}^{(1)} \ll E_J^{(1)}, \qquad E_{C_m}^{(2)} \gg E_J^{(2)} \tag{60}$$

We start by performing a vortex integration in array 2. This can be done using the Poisson summation formula to write

$$\sum_{\{v^{(2)}\}} \exp[-S^{(2)}(v^{(1)})] = \sum_{\{P\}} \int \prod_{\tau,\vec{r}} d\Phi(\tau,\vec{r}) \exp[-S^{(2)}(\Phi)] \exp\left[2\pi i \sum_{\tau,\vec{r}} \Phi(\tau,\vec{r})\, P(\tau,\vec{r})\right] \tag{61}$$

We can neglect the $P \neq 0$ terms when $E_{C_m}^{(2)} \gg E_J^{(2)}$, since they are exponentially small. Note that this integration only involves vortices in array 2, and it can be done without affecting the variables in array 1. When we only consider the $P = 0$ terms, there is a change in the Josephson coupling constant given by $E_J^{(2)} \to E_J^{(2)}/2$.[12] In this parameter limit we see that the only modification is in the new coupling constant in Eq. (59).

We can also use the Poisson summation formula for the integration of the charges in array 1. After the integration the action for the charges can be written as

$$S^{(1)}[n^{(1)}] = \tfrac{1}{2} \sum_{\tau,\tau',\vec{r}_1,\vec{r}_2} n^{(1)}(\tau,\vec{r}_1)\, \mathbf{M}(\tau,\tau';\vec{r}_1,\vec{r}_2)\, n^{(1)}(\tau',\vec{r}_2) + \sum_{\tau,\vec{r}} J(\tau,\vec{r})\, n^{(1)}(\tau,\vec{r}) \tag{62}$$

where the $\mathbf{M}$ operator and effective current $J$ are

$$\mathbf{M}(\tau,\tau';\vec{r}_1,\vec{r}_2) = \frac{q^2\varepsilon}{\pi} \tilde{\mathbf{C}}_{1,1}(\vec{r}_1,\vec{r}_2)\, \delta_{\tau,\tau'} - \frac{1}{2\pi\varepsilon E_J^{(1)}} \mathbf{G}(\vec{r}_1,\vec{r}_2)\, \Delta_\tau^2 \tag{63}$$

$$J(\tau,\vec{r}_1) = \sum_{\vec{r}_2} \left[ i\Theta(\vec{r}_1,\vec{r}_2)\, \Delta_\tau v^{(1)}(\tau,\vec{r}_2) + \frac{q^2\varepsilon}{\pi} \tilde{\mathbf{C}}_{1,2}(\vec{r}_1,\vec{r}_2)\, n^{(2)}(\tau,\vec{r}_2) \right] \tag{64}$$

After integrating the charges in array 1 and the vortices in array 2 we are left with the following expression for the effective partition function

$$Z = \sum_{\{v^{(1)}\}} \sum_{\{n^{(2)}\}} \exp[-S_{\text{eff}}(v^{(1)}, n^{(2)})] \tag{65}$$

where the effective action for vortices in array 1 and charges in array 2 is given by

$$\begin{aligned} S_{\text{eff}}[v^{(1)}, n^{(2)}] = \sum_{\tau, \vec{r}_1, \vec{r}_2} \Bigg[ & \tau\varepsilon E_J^{(1)} v^{(1)}(\tau, \vec{r}_1)\, \mathbf{G}(\vec{r}_1, \vec{r}_2)\, v^{(1)}(\tau, \vec{r}_2) \\ & + \frac{1}{2\tau\varepsilon E_J^{(2)}} \Delta_\tau n^{(2)}(\tau, \vec{r}_1)\, \mathbf{G}(\vec{r}_1, \vec{r}_2)\, \Delta_\tau n^{(2)}(\tau, \vec{r}_2) \Bigg] \\ & + \sum_{\tau, \tau', \vec{r}_1, \vec{r}_2} \Bigg[ \frac{\varepsilon}{2} n^{(2)}(\tau, \vec{r}_1)\, \mathbf{G}_{\mathbf{n}}(\tau, \tau'; \vec{r}_1, \vec{r}_2)\, n^{(2)}(\tau', \vec{r}_2) \\ & + i n^{(2)}(\tau, \vec{r}_1)\, \tilde{\boldsymbol{\Theta}}(\tau, \tau'; \vec{r}_1, \vec{r}_2)\, \Delta_\tau v^{(1)}(\tau', \vec{r}_2) \\ & + \frac{\pi}{2q^2\varepsilon} \Delta_\tau v^{(1)}(\tau, \vec{r}_1)\, \mathbf{G}_{\mathbf{v}}(\tau, \tau'; \vec{r}_1, \vec{r}_2)\, \Delta_\tau v^{(1)}(\tau', \vec{r}_2) \Bigg] \end{aligned} \tag{66}$$

The effective interaction potentials are defined by

$$\begin{aligned} \mathbf{G}_{\mathbf{n}}(\tau, \tau'; \vec{r}_1, \vec{r}_2) = \frac{q^2}{\pi} \Bigg[ & \tilde{\mathbf{C}}_{2,2}(\vec{r}_1, \vec{r}_2)\, \delta_{\tau,\tau'} - \frac{q^2\varepsilon}{\pi} \sum_{\vec{r}_3, \vec{r}_4} \tilde{\mathbf{C}}_{1,2}(\vec{r}_1, \vec{r}_3) \\ & \times \mathbf{M}^{-1}(\tau, \tau'; \vec{r}_3, \vec{r}_4)\, \tilde{\mathbf{C}}_{1,2}(\vec{r}_4, \vec{r}_2) \Bigg] \end{aligned} \tag{67}$$

$$\tilde{\boldsymbol{\Theta}}(\tau, \tau'; \vec{r}_1, \vec{r}_2) = -\frac{q^2\varepsilon}{\pi} \sum_{\vec{r}_3, \vec{r}_4} \boldsymbol{\Theta}(\vec{r}_2, \vec{r}_3)\, \mathbf{M}^{-1}(\tau, \tau'; \vec{r}_3, \vec{r}_4)\, \tilde{\mathbf{C}}_{2,1}(\vec{r}_4, \vec{r}_2) \tag{68}$$

$$\mathbf{G}_{\mathbf{v}}(\tau, \tau'; \vec{r}_1, \vec{r}_2) = -\sum_{\vec{r}_3, \vec{r}_4} \boldsymbol{\Theta}(\vec{r}_1, \vec{r}_3)\, \mathbf{M}^{-1}(\tau, \tau'; \vec{r}_3, \vec{r}_4)\, \boldsymbol{\Theta}(\vec{r}_4, \vec{r}_2) \tag{69}$$

Notice that the time nonlocality of these kernels comes from the second term in Eq. (63). To gain some physical understanding of these complicated equations we will next discuss a simplification when the nonlocal term in Eq. (63) is small.

## A. Vortex-Charge Capacitive Gauge-like Coupling

In Eq. (66) we have an effective interaction between vortices in array 1 and charges in array 2. We consider the dynamics of just one charge and

one vortex in each array, in a similar way as was done for one array in ref. 12. Lets assume that the vortex and the charge move along the imaginary time-dependent trajectories $\vec{R}(\tau)$, and $\vec{X}(\tau)$ respectively. In this case the vortex and charge space-time distributions can be described by

$$v^{(1)}(\tau, \vec{r}_1) = \delta[\vec{r}_1 - \vec{R}(\tau)]; \qquad n^{(2)}(\tau, \vec{r}_2) = \delta[\vec{r}_2 - \vec{X}(\tau)] \tag{70}$$

After taking the time derivative of $v^{(1)}$ we find

$$\Delta_\tau v^{(1)}(\tau, \vec{r}) = \Delta_\tau \delta[\vec{r}_1 - \vec{R}(\tau)] = -\sum_\mu \Delta_\mu \delta[\vec{r}_1 - \vec{R}(\tau)]\, \Delta_\tau \vec{R}(\tau) \tag{71}$$

The right hand side in this equation relates the time derivative to a summation over space derivatives. We can next rewrite the interaction term in the effective action in the following way

$$S_{\text{int}} = -i \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \sum_\mu \Delta_\mu \tilde{\Theta}(\tau, \tau'; \vec{X}(\tau), \vec{R}(\tau')) \frac{dR_\mu(\tau')}{d\tau} \tag{72}$$

This expression has a similar form to the typical minimal gauge coupling in electrodynamics. Following this analogy we can define the corresponding vector potential

$$\vec{A}(\tau', \vec{R}(\tau')) = \int_0^{\beta\hbar} d\tau\, \vec{\Delta}\tilde{\Theta}(\tau, \tau'; \vec{r}, \vec{R}(\tau')) \tag{73}$$

Using this definition Eq. (72) can be rewritten as

$$S_{\text{int}} = -i \int_0^{\beta\hbar} d\tau'\, \vec{A}(\tau', \vec{R}(\tau')) \cdot \frac{d\vec{R}(\tau')}{d\tau} \tag{74}$$

Here we have chosen to view the vortex-charge interaction in the representation where the vortex moves under the influence of the charge gauge-like field $\vec{A}$. This view is equivalent to the representation where the charge moves in the gauge-like field produced by the vortex. We have a vector field, and we can find its corresponding effective "magnetic field"

$$\vec{B}(\tau, \vec{r}) = \vec{\Delta} \times \vec{A}(\tau, \vec{r}),$$

$$= -\frac{q^2}{\pi} \int_0^{\beta\hbar} d\tau' \sum_{\vec{r}_3, \vec{r}_4} (\vec{\Delta} \times \vec{\Delta}\Theta(\vec{X}(\tau'), \vec{r}_3))\, \mathbf{M}^{-1}(\tau, \tau'; \vec{r}_3, \vec{r}_4)\, \tilde{\mathbf{C}}_{1,2}(\vec{r}_4, \vec{r}) \tag{75}$$

The solution kernel $\Theta$ for a point vortex at the origin satisfies the equation

$$\vec{\Delta} \times \vec{\Delta}\Theta(\vec{r}_1, \vec{r}_2) = 2\pi\delta_{\vec{r}_1, \vec{r}_2}\hat{k} \tag{76}$$

from which we can get our final expression for the effective magnetic field

$$\vec{b}(\tau, \vec{r}) = -\frac{q^2\varepsilon}{\pi}\int_0^{\beta\hbar} d\tau' \sum_{\vec{r}_1} \mathbf{M}^{-1}(\tau, \tau'; \vec{X}(\tau'), \vec{r}_1)\, \tilde{\mathbf{C}}_{1,2}(\vec{r}_1, \vec{r})\, \hat{k} \tag{77}$$

Up to now we have that the effective action in Eq. (66), the effective gauge vector potential and its magnetic field interaction are nonlocal in time, due to the nonlocality in time of the second term in Eq. (63). In the limit $(\beta E_J^{(1)})(\beta E_{C_m}^{(1)}) \gg 1$, the second term in Eq. (63) is negligible, and therefore we can write

$$\mathbf{M}^{-1}(\tau, \tau'; \vec{r}_1, \vec{r}_2) \approx \frac{\pi}{q^2\varepsilon}\tilde{\mathbf{C}}_{1,1}^{-1}(\vec{r}_1, \vec{r}_2)\, \delta_{\tau, \tau'} \tag{78}$$

Using Eq. (8) we can write this equation in terms of the intra-array capacitance matrix

$$\tilde{\mathbf{C}}_{1,1}^{-1} = [\mathbf{C}_1 - C_{\text{int}}^2 \mathbf{C}_2^{-1}] \tag{79}$$

We note that the array is periodic and symmetric, so that all the commuting matrix operators can be diagonalized using plane waves. These facts produce important simplifications in the rest of the interaction kernels, giving the results

$$\tilde{\Theta} \approx -C_{\text{int}}\Theta\mathbf{C}_2^{-1}\, \delta_{\tau, \tau'} \tag{80}$$

$$\mathbf{G}_n \approx \frac{q^2}{\pi}\mathbf{C}_2^{-1}\, \delta_{\tau, \tau'} \tag{81}$$

$$\mathbf{G}_v \approx \Theta[\mathbf{C}_1 - c_{\text{int}}\mathbf{C}_2^{-1}]\, \Theta\, \delta_{\tau, \tau'} \tag{82}$$

Equation (81) is particularly significant, since it implies that, within this approximation, the interactions among charges in array 2 do not depend on the presence of array 1. This is a counter-intuitive result, because we would expect that a virtual photon excited from an island in array 2 and absorbed in another island in the same array would have contributions from bounce interactions with array 1. What happens is that after adding all the contributions from these bounces, the net result (within the

approximation leading to Eq. (78)) is a cancellation of the contributions arising from array 1. Finally, we can write the effective magnetic field as

$$\vec{b}(\vec{R}(\tau)) \approx -C_{\text{int}} \mathbf{C}_2^{-1}(\vec{r}, \vec{R}(\tau))\,\hat{k} \tag{83}$$

This result implies that, if we have a charge at $\vec{X}(\tau)$ and a vortex at $\vec{R}(\tau)$, the vortex will feel an effective magnetic field produced by the charge of magnitude $-C_{\text{int}}\mathbf{C}_2^{-1}(\vec{R}(\tau), \vec{X}(\tau))$. This situation is reminiscent of the vortex-charge bound states extensively studied in the fractional quantum Hall effect problem, and it may very well be that this system may serve as an experimental prototype for those types of problems.

Our discussion here has concentrated on deriving and analyzing convenient partition function expressions that one can also use in quantum Monte Carlo simulations. We have done some work in this direction, but we must say that the problem is still highly non trivial because of the form of the kernels in the effective action. However, we expect to further unravel interesting physics for this problem in future.

## V. CONCLUSIONS

In this paper we have introduced and presented results for a model of two capacitively coupled quantum Josephson junction arrays. This is a difficult problem but one that promises to lead to interesting new physics. We have first derived a semiclassical expression for an effective Hamiltonian, that allowed us to study the change in the critical temperature for each array. Vile used two types of variational actions that permitted the evaluation of critical temperature shifts as a function of the inter-layer coupling capacitance. The main qualitative result is that an increase in the interaction capacitance increases phase coherence in the arrays. Next we considered the interesting case where one array is quantum phase dominated and the other Cooper pair charge dominated. Here we extended the one-array work of Fazio *et al.*[12] to the capacitively coupled two-array problem. We wrote an effective action in terms of four interacting imaginary time Coulomb-like gases, and derived an effective Hamiltonian for the coupled system. The effective Hamiltonian is dually symmetric between charge and vorticity in form but with complicated kernels. In the simplified case where one array has one vortex and the other one charge, we showed that their interaction has a minimal gauge-like coupling. This interaction is, however, nonlocal in the gauge field.

Finally, this type of system holds the promise to lead to a variety of novel experimentally observable macroscopic quantum phenomena. In particular, the vortex-charge interaction discussed at the end of this paper

deals with the interplay of quantum-classical effects, and may lead to possible fractional statistics analogies to the fractional quantum Hall effect.

## ACKNOWLEDGMENTS

I am delighted to be able to add this contribution in honor of Leo Kadanoff's sixtieth birthday. I have purposefully chosen a subject that involves *duality transformations*. Although in a completely different context, this goes back to several ideas I first learned from Leo many years ago, when I started my research career under his tutelage. Duality transformations continue to lead to new and interesting physics. Leo has been involved with *duality transformations* in several different contexts, making seminal contributions to their understanding and applications.

I spent about three years with Leo, two as a his Ph.D. graduate student and one as his postdoctoral fellow. During this time I was able to see a real Master at work. Although we did not meet very often, every time we did (and he was always available to meet me), his insights and quickness left a deep impression on me. Everybody has his/her own style of doing research, and while I could not claim to do research as Leo does, his approach, style and his depth made a lasting impression on my way of looking at physics and physics problems addressing their detailed quantitative solutions. This work has been partially supported by NSF Grant DMR-9521845.

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